Groups of order 4

Suppose G is a group with $|G|=4$.

Then it turns out that G is isomorphic to:

- C_4, or
- The Klein 4-group:

(Note: $C_4 \not\cong V_4$)

$$V_4 = \{e, a, b, \underset{ab}{\underset{\shortparallel}{c}}\}$$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Note:
- V_4 is Abelian
- $c = ab$
- $a^2 = b^2 = c^2 = e$

(2 groups of order 4, up to isomorphism)

Another way to write this group:

$$V_4 = \langle \underbrace{a, b}_{\text{(generators)}} \mid \underbrace{a^2 = b^2 = e,\ ab = ba}_{\text{(relations)}} \rangle$$

(a presentation for the group)

This notation indicates that:

i) a and b generate the group: every element of the group is a finite product of a's and b's. (and their inverses)

ii) a and b satisfy the relations indicated, and any other relation satisfied by elements of the group can be deduced from these.

Ex: $(ab)^2 = (ab)(ab)$

$= a(ba)b$ (gen. assoc.)

$= a(ab)b$ (relation: $ab = ba$)

$= a^2b^2$ (gen. assoc.)

$= ee$ (relations: $a^2 = e$, $b^2 = e$)

$= e.$

And another way to write this group...

Direct products

If $(G,*)$ and $(H,\circ)$ are groups then their direct product is the group $(G\times H, \cdot)$ defined by:

$$(g_1,h_1)\cdot(g_2,h_2) = (g_1 * g_2, h_1\circ h_2), \quad \forall (g_1,h_1),(g_2,h_2)\in G\times H.$$

Notation: $G\times H$

Check that this is a group: (supress notation for binary ops.)

- associativity:

$$\begin{aligned}
((g_1,h_1)(g_2,h_2))(g_3,h_3) &= (g_1g_2, h_1h_2)(g_3,h_3) \\
&= ((g_1g_2)g_3, (h_1h_2)h_3) \\
&= (g_1(g_2g_3), h_1(h_2h_3)) \quad \text{(assoc. in } G \text{ and } H\text{)} \\
&\;\vdots \\
&= (g_1,h_1)((g_2,h_2)(g_3,h_3))
\end{aligned}$$

- identity $= (e_G, e_H)$:

$\forall (g,h)\in G\times H$,

$$(g,h)(e_G,e_H) = (ge_G, he_H) = (g,h)$$

$$(e_G,e_H)(g,h) = (e_G g, e_H h) = (g,h)$$

• inverse of (g,h) is (g^{-1}, h^{-1}):

$$(g,h)(g^{-1},h^{-1}) = (gg^{-1}, hh^{-1}) = (e_G, e_H)$$
$$(g^{-1},h^{-1})(g,h) = (g^{-1}g, h^{-1}h) = (e_G, e_H)$$

Ex:

$G = C_2 = \langle x \rangle = \{e_G, x\}$ $H = C_2 = \langle y \rangle = \{e_H, y\}$

$$G \times H = \{\underbrace{(e_G,e_H)}_{e}, \underbrace{(x,e_H)}_{a}, \underbrace{(e_G,y)}_{b}, \underbrace{(x,y)}_{c}\}$$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Conclusion: $V_4 \cong C_2 \times C_2$.

More generally, if $G_1, \dots, G_n$ are groups then their direct product $G_1 \times \cdots \times G_n$ is their Cartesian product, together with the binary operation defined by:

$$(g_1,\dots,g_n)(g_1',\dots,g_n') = (\underbrace{g_1g_1'}_{\substack{\text{bin. op.}\\ \text{in } G_1}}, \underbrace{g_2g_2'}_{\substack{\text{bin. op.}\\ \text{in } G_2}}, \dots, \underbrace{g_ng_n'}_{\substack{\text{bin. op.}\\ \text{in } G_n}}).$$

. . .

Groups of order 5

Suppose G is a group with $|G|=5$

Then $G \cong C_5$.

General fact (will prove later):

If G is a group with $|G|=p$, where p is prime, then $G \cong C_p$.

Groups of order 6

Suppose G is a group with $|G|=6$

Then $G \cong C_6$ or $G \cong D_6$. (Note: $C_6 \not\cong D_6$)

(dihedral group of order 6)

"Obvious" questions:

1) What is D_6? (wait a few minutes)

2) Why isn't $C_2 \times C_3$ on the list above?

Answer: $C_2 \times C_3 \cong C_6$.

To see this, write ($C_2 = \langle x \rangle$, $C_3 = \langle y \rangle$)

$$C_2 \times C_3 = \{(e_2,e_3), (e_2,y), (e_2,y^2), (x,e_3), (x,y), (x,y^2)\},$$

and note that

$$(x,y)^1 = (x,y)$$
$$(x,y)^2 = (x^2,y^2) = (e_2,y^2)$$
$$(x,y)^3 = (e_2,y^2)(x,y) = (x,y^3) = (x,e_3)$$
$$(x,y)^4 = (x^4,y^4) = (e_2,y)$$
$$(x,y)^5 = (x^5,y^5) = (x,y^2)$$
$$(x,y)^6 = (x^6,y^6) = (e_2,e_3).$$

Therefore $C_2 \times C_3 = \langle (x,y) \rangle \cong C_6$.

(symmetric group of degree 3)

3) Why isn't S_3 on the list above?

Answer: $S_3 \cong D_6$. (details about symmetric groups in later videos)

Dihedral groups

Let $n \geq 3$. The collection of all rigid motions preserving a regular n-gon centered at the origin in the plane, with the binary operation of composition of maps, forms a group, D_{2n}, called the <u>dihedral group of order $2n$</u>.

Note: "rigid motions" = "isometries" (preserve distances between points)

$= \begin{cases} \cdot \text{ translations} \\ \cdot \text{ rotations} \\ \cdot \text{ reflections} \\ \cdot \text{ compositions of these} \end{cases}$

Check that D_{2n} is a group:

- Associativity ✓ (Comp. of maps $f: S \to S$ is associative)
- Identity: ✓

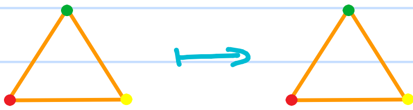
  $e: \mathbb{R}^2 \to \mathbb{R}^2$
  $(x,y) \mapsto (x,y)$

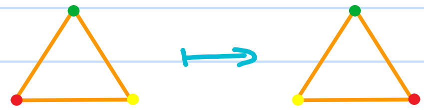
- Inverses: Every rigid motion $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection, and the inverse function $f^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ is also a rigid motion, which satisfies $f \circ f^{-1} = f^{-1} \circ f = e$. ✓

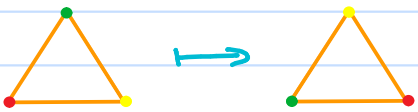
Exs:

1) $n=3$

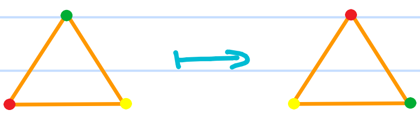
(Reminder about compositions of maps: To figure out where they map points, work from right to left.)

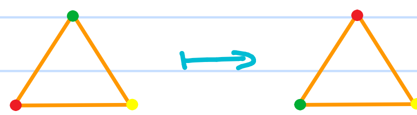
e : 

s : 

r : 

$s \circ r = sr$: 

$r \circ r = r^2$: 

$s \circ r^2 = sr^2$: 

$$D_6 = \{e, r, r^2, s, sr, sr^2\}$$

Note: $r^3 = e$, $\quad (r^2 r = e \Rightarrow r^{-1} = r^2)$

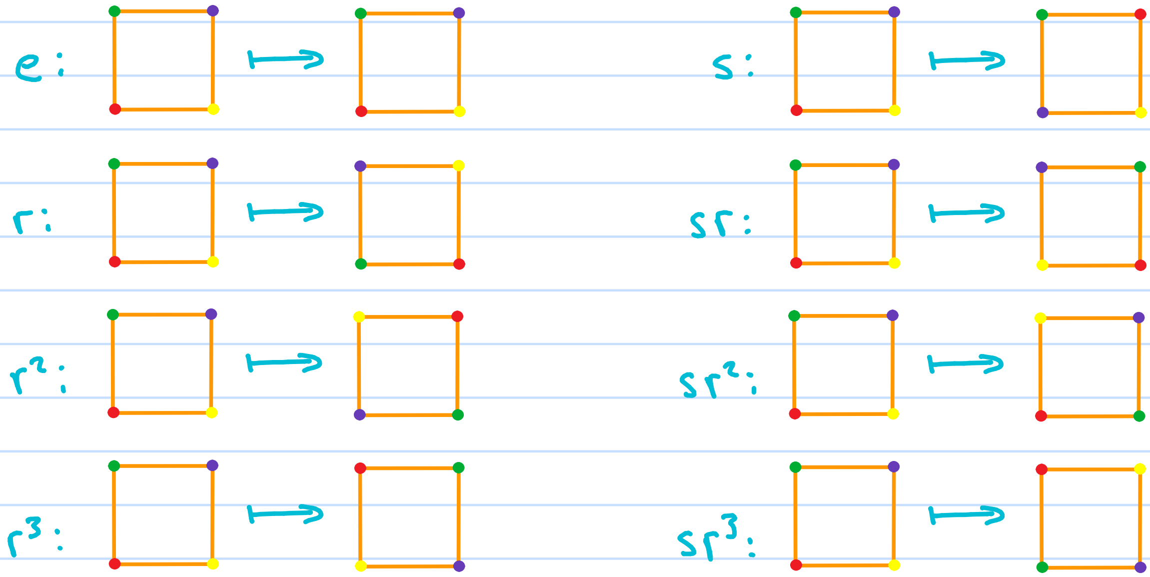
$s^2 = e$, $\quad (s^{-1} = s)$

$rs = sr^2 = sr^{-1}$

$$D_6 = \langle r, s \mid r^3 = s^2 = e,\ rs = sr^{-1} \rangle$$

Also: $rs \neq sr \Rightarrow$ <u>D_6 is non-Abelian</u>

2) $n=4$



$$D_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$$
$$= \langle r, s \mid r^4 = s^2 = e,\ rs = sr^{-1} \rangle$$

3) In general, for $n \geq 3$:

(rotation by $2\pi/n$) (reflection about line through origin and fixed angle)

$$D_{2n} = \{e, r, r^2, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\} \quad (|D_{2n}| = 2n)$$

$$= \langle r, s \mid r^n = s^2 = e,\ rs = sr^{-1} \rangle$$

($rs = sr^{-1} = sr^{n-1} \neq sr \Rightarrow D_{2n}$ is non-Abelian)

Groups of order 7

Suppose G is a group with $|G| = 7$

Then $G \cong C_7$. (7 is prime - see fact from before)

Groups of order 8

Suppose G is a group with $|G| = 8$.
Then G is isomorphic to one of the following five (non-isomorphic) groups:

$$\underbrace{C_8,\ C_2 \times C_4,\ C_2 \times C_2 \times C_2}_{\text{Abelian}},\ \underbrace{D_8,\ Q_8}_{\text{non-Abelian}}.$$

(quaternion group)

The <u>quaternion group</u> Q_8 is

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\},$$

with (multiplicative) binary operation determined by

$1 \cdot x = x \cdot 1 = x, \quad \forall x \in Q_8$ (1 is the identity)

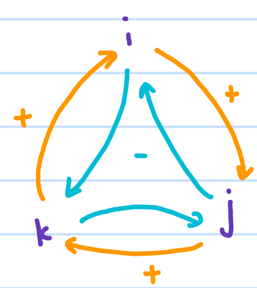
$(-1) \cdot x = x \cdot (-1) = -x, \quad \forall x \in Q_8$

$i^2 = j^2 = k^2 = -1$

$ij = k \qquad ji = -k$

$jk = i \qquad kj = -i$

$ki = j \qquad ik = -j$



Ex: $(-i)^2 = (-i)(-i) = (i(-1))((-1)i) = i\underbrace{((-1)(-1))}_{=1}i = i^2 = -1$

Similarly: $(-j)^2 = (-k)^2 = -1$

Summary: Groups of orders $1 \leq n \leq 8$, up to isomorphism.

n	
1	C_1
2	C_2
3	C_3
4	C_4, $V_4 \cong C_2 \times C_2$
5	C_5
6	$C_6 \cong C_2 \times C_3$, $D_6 \cong S_3$
7	C_7
8	C_8, $C_2 \times C_4$, $C_2 \times C_2 \times C_2$, D_8, Q_8

$\blacksquare$ = Abelian

$\blacksquare$ = non-Abelian